On the decomposition of the Gini index: An exact approach

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Abstract

The decomposition of inequality indices across household groups is useful to judge the importance of contribution of each group to the total inequality. The decomposition of relative indices of inequality, such as the Gini index, is not a simple procedure because that this index is not additively separable according to the contribution of the basic units that are incomes or standard living of households. In this paper, we propose methods which are based on the ethical values, that this index respects, to perform an exact algebraic decomposition, and that, for a better interpretation of the decomposition components. Also, we propose the use of the Shapley approach to perform the decomposition according to household groups or sources of income.

Mots clés: Equity, Inequality, Decomposition, Shapley.
Classification JEL: D63, D64.

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1 Introduction

The decomposition of the Gini index in inequality inter and intragroup raises a quite founded concern. Indeed, the decomposition of this index can generate a residue which is not simple to interpret. Generally, when it is supposed that the intergroup inequality is that where each household has the average income of its group, the algebraic decomposition of the Gini index takes the following form:\(^1\):

\[ I = \sum_g a_g I_g + \bar{I} + R \]  

(1)

where \( \bar{I} \) is the component of the intergroup inequality, \( a_g \) is the product between the proportion of group \( g \) and the ratio between the mean of income of the group and its of the population. The component \( R \) denote the residue.

Let us recall here that the Gini index can be written in several algebraic forms. For each one of these forms, one can give an interpretation or a definition to this index. In our approach of decomposition, one wants that the components of decomposition keeps the same interpretation at the level of the component. Also, Which distinguishes our approach, is the definition of the intergroup inequality. We take into account the level of household income instead of supposing that the household has the average income of its group. As we will see it, that made that our algebraic decomposition is exact and is written in the following form:

\[ I = \sum_g \hat{a}_g \hat{I}_g + \tilde{I} \]  

(2)

2 Decomposition of the Gini index and relative deprivation

According to Runciman (1966)\(^4\), the magnitude of relative deprivation is the extent of the difference between the desired situation and that of the person desired it. We define the relative deprivation of household \( i \) compared to \( j \) as follows\(^2\):

\[ \delta_{i,j} = (y_j - y_i)_+ = \begin{cases} y_j - y_i & \text{if } y_i < y_j \\ 0 & \text{otherwise} \end{cases} \]  

(3)

\(^1\)See also the interpretation of P. Lambert and R. Aronson (1993)\(^3\).
\(^2\)See also Yizhaki (1979)\(^7\) and Hey and Lambert (1980)\(^2\)
The expected deprivation of household \(i\) is equals to what follows:

\[
\bar{\delta}_i = \frac{\sum_{j=1}^{N} (y_j - y_i)_+}{N}
\]  

(4)

The Gini index can be write by the following form:

\[
I = \sum_{i=1}^{N} \bar{\delta}_i \mu y_i N
\] 

(5)

This form of writing of the coefficient of Gini shows that this coefficient is only the average of the expected relative deprivation \(\bar{\delta}\), divided by the average of the incomes. The contribution of each household or individual to the coefficient of Gini is then its average relative deprivation, noted by \(\bar{\delta}_i\). When the household \(k\) belongs to group \(g\), its average relative deprivation can be also written, in the following form:

\[
\bar{\delta}_k = \phi_g \bar{\delta}_{k,g} + \tilde{\delta}_{k,g}
\] 

(6)

\[
\tilde{\delta}_{g,k} = \frac{\sum_{j=1}^{N-K_g} (y_k - y_j)_+}{N} 
\] 

(7)

where \(\tilde{\delta}_{k,g}\) represents the average relative deprivation compared to households who do not belong to the group \(g\). By rewriting the Gini index, one will have as follows:

\[
I = \sum_{g=1}^{G} \sum_{k=1}^{K_g} \left[ \frac{\phi_g \bar{\delta}_{k,g} + \tilde{\delta}_{k,g}}{\mu N} \right]
\] 

\[
= \sum_{g=1}^{G} \left[ \frac{\phi_g^2 \mu g K_g}{\mu N g_k} \right] + \sum_{i=1}^{N} \frac{\bar{\delta}_i}{\mu N}
\] 

\[
= \sum_{g=1}^{G} \phi_g^2 \mu g I_g + \bar{I}
\] 

(10)
where \( \tilde{I} \) equals to the Gini index when the relative deprivation to the group is ignored. In this direction, for the intergroup inequality, one is interested only in the relative deprivation compared to households that represents the complement of the group.

We can note here, in the case of income distribution without overlap, we write the decomposition of the Gini index as follows:

\[
G = \sum_{g=1}^{G} \phi^2_g \frac{\mu_g}{\mu} I_g + I(\mu_k)
\]  

(11)

Recall here, when overlaps exist, some members of the richer group are poorer than some members of the poor group. In this special case, the approach that we propose for our vision with inter group inequality and what is proposed in later works are conciliates.

In this first decomposition, we concentrated on the fact that the index of Gini is based on relative deprivation. We go, in the following section, to reformulate this index and to write it in a form which holds account the rank or the classification of households according to income level.

### 3 Decomposition of single parameterized Gini index

Donaldson and Weamark (1980)\footnote{Voir Donaldson et Weamark (1980)} propose to generalize the Gini index of inequality. The single parameterized Gini index depends on the ethical parameter, noted by \( \rho \), that express the level of aversion to inequality of the society. This index takes the following form:

\[
I_\rho = 1 - \frac{\xi_\rho}{\mu_y}
\]

(12)

\footnote{In the case of distribution without overlap, the relative deprivation of a given member of poor group to others \( m \) members of reach group is equivalent to the \( m \) difference between the mean of the reach group and its revenue. Also comparing between the revenues of \( n \) members of poor group and others reach members is equivalent to use its mean.}
where \( \xi_\rho = \sum_{i=1}^{N} \frac{\nu(i, \rho)}{N^\rho} y_i \), \( \nu(i, \rho) = i^\rho - (i-1)^\rho \) and \( y_1 \geq y_2 \geq \cdots \geq y_N \)

One can write also:

\[
\xi_\rho = \sum_{i=1}^{N} p_{i, \rho} y_i \quad (13)
\]

For the ordinary Gini index, the parameter \( \rho \) equals to 2 and \( \nu(i) = (2i - 1)/N^2 \). We can note here, when \( \rho > 1 \), the weight \( p_{i, \rho} \) decrease sharply with the rank of the household. In Another manner, the weight attributed the poorest household is greatest rel ativelly to the richest household. Despite the fact that the weight \( p_i \) depends on the rank of household \( i \), the social welfare function is additively separable on incomes. Hence, we can rewrite this function by using notation at the level of groups such as:

\[
\xi_\rho = \sum_{g=1}^{G} \sum_{k=1}^{K_g} p_{g, k, \rho} y_k \quad (14)
\]

where \( G \) is the number of groups and \( K_g \) the number of households which belong to the group \( g \). The weight \( p_{g, k, \rho} \) is the same that is attributed to the household \( k \) for the social welfare at population level. By rewriting the parameterized Gini index, we have:

\[
I_\rho = \sum_{g=1}^{G} \left[ \Phi_g - \frac{\tilde{\xi}_{g, \rho}}{\mu} \right] \quad (15)
\]

where \( \Phi_g = \phi_g \mu_g \) and \( \tilde{\xi}_{g, \rho} = \sum_{k=1}^{K} p_{g, k, \rho} y_k \). The parameter \( \phi_g \) represents the proportion of groupe \( g \), \( \mu \) and \( \mu_g \)represent respectively the mean of incomes of the population and its of the group \( g \). The exact contribution of group \( g \) is given by:

\[
C_g = \Phi_g - \frac{\tilde{\xi}_{g, \rho}}{\mu} \quad (16)
\]

It is clear that, according to equation \( 13 \) the weight \( p_i \), attributed to household \( i \) to compute in inequality at the population level, will be different from this attributed to compute inequality at the group level. By rewriting the contribution of group \( g \) to the function \( \xi \), that define the social welfare of total population, we have that as follows:
\[ \tilde{\xi}_{g,\rho} = \sum_{k=1}^{K_g} p_{g,k,\rho} y_{g,k} \]  
\[ = \sum_{k=1}^{K_g} \left( \phi^g_{\rho} \left( \pi_{g,k,\rho} + \tau_{g,k,\rho} \right) \right) y_{g,k} \]  
\[ = \phi^g_{\rho} \left( \xi_{g,\rho} + \xi^*_{g,\rho} \right) \]  

where \( \pi_{g,k,\rho} \) is the weight attributed for the social welfare function at the level of the group for household \( k \) and \( \tau_{g,k,\rho} \) represents the reclassification impact in presence of the others groups. If the others groups does not exist we have then \( \tau_{g,k} = 0 \). By using the last equation, one can write the Gini index as follows:

\[ I_\rho = \sum_{g=1}^{G} \left[ \frac{\mu_g}{\mu} \left( \phi^g_{\rho} - \frac{\tilde{\xi}_{g,\rho,\mu}}{\mu_g} \right) \right] \]  
\[ = \sum_{g=1}^{G} \left[ \frac{\mu_g}{\mu} \phi^g_{\rho} \left( 1 - \frac{\tilde{\xi}_{g,\rho,\mu}}{\mu_g} \right) \right] + 1 - \frac{\sum_{g=1}^{G} \phi^g_{\rho} \left( \xi^*_{g,\rho} + \mu_g \right)}{\mu} \]  
\[ = \sum_{g=1}^{G} \phi^g_{\rho} \frac{I_{g,\rho}}{\mu} I + \tilde{I} \]

For the ordinary Gini index, the decomposition can be writhe as follows:

\[ I = \sum_{g=1}^{G} \phi^g_{\rho} \frac{I_{g,\rho}}{\mu} I + \tilde{I} \]

Such as \( \tilde{I} \) can be perceived like the term representing the intergroup inequality. To notice here that the residue does not appear, and that, is simply with our vision of what really represent the intergroup inequality. As the index of Gini is based, a priori, on the rank of households, the impact of the other groups on the contribution of household, is simply with its reclassification after having held account households of the other groups. Again one arrives at the same result expressed by the equation \( 10 \) but with a different interpretation of the intergroup component.
Let us recall here that, the basic approach consists in measuring the intergroup inequality when the standard living of each household is equal to the average standards living of its group. The new approach that we propose, is based rather on the additional impact of the existence of households of the other groups on the contribution of the household. This approach makes that the residue does not appear in this type of decomposition. For better determining this idea, we will discuss the inequality will intra and intergroup for the example presented in following table:

<table>
<thead>
<tr>
<th>Ménage</th>
<th>A</th>
<th>B</th>
<th>B’</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>11</td>
<td>19</td>
</tr>
</tbody>
</table>

In this example, one supposes that the population consists of two groups, A and B, and each group are composed of two households. As, it is supposed as B’ represents a potential distribution of incomes for the group B. If one bases oneself on the average of incomes at the level of groups to define the intergroup inequality, this inequality is the same one for the two cases, that is to say B and B’. However, if one bases oneself on the new approach which we propose, this intergroup inequality is not the same one since we take account of the relative deprivations of the intergroup households.

4 Decomposition of the Gini index according to the Shapley approach

4.1 The Shapley approach

Applied in several scientific domains, the Shapley approach can serve to perform an exact decomposition of the distributive indices, such as the Gini index.\(^5\)

The Shapley value is a solution concept often employed in the theory of cooperative games. We consider a set \(N\), constituted of \(n\) players that must divide a given surplus among themselves. The players group themselves together to form coalitions (these are the subsets \(S\) of \(N\)) that appropriate themselves a part of the surplus and redistribute it between their members. We suppose that the function \(v\) determines the coalition force, i.e., which surplus will be divided without resorting

\(^5\)Voir Shorrocks (1999) [6].
to an agreement with the other players (that are members of the complementary coalition: \((n - s - 1)\) players). The question to resolve is the following one: how can we divide the surplus between the \(n\) players? According to the Shapley approach, introduced by Lloyd Shapley in 1953[5], the value of the player \(k\) in the game, given by the characteristic function \(v\), is given by the following formula:

\[
C_k = \sum_{s \in \{0, n-1\}} \frac{s!(n-s-1)!}{n!} \text{MV}(S, k)
\]

(24)

The term, \(\text{MV}(S, k)\) is equal to the marginal value that the player \(k\) generates after his adhesion to the coalition \(S\). What will then be the expected marginal contribution of player \(k\), according to the different possible coalitions that can be formed and to which the player can adhere? First of all, the size of the coalition \(S\) is limited such as: \(s \in \{0, 1, \ldots, n-1\}\). Suppose that the \(n\) players are randomly ordered according to an order, noted by \(\sigma\), such as:

\[
\sigma = \left\{ \sigma_1, \sigma_2, \ldots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \ldots, \sigma_n \right\}
\]

(26)

For each of the possible permutations of the \(n\) players, which number \(n!\), the number of times that the same first \(s\) players are located in the subset or coalition \(S\) is given by the number of possible permutation of the \(s\) players in coalition \(S\), that is, \(s!\). For every permutation in the coalition \(S\), we find \((n-s-1)!\) permutations for the players that complement the coalition \(S\).

The expected marginal value that player \(k\) generates after his adhesion to a coalition \(S\) is given by the Shapley value. For every position of the factor \(k\) (predetermined cuts of the coalition \(S\)), there are several possibilities to form coalitions \(S\) from the \(n - 1\) player (the \(n\) players without the player \(k\)). This number of possibilities is equal to the combination \(\binom{n}{n-1}\).

How many marginal values would one have to calculate to calculate the expected marginal contribution of a given factor, be the factor \(k\)? Because the order of the players in the coalition \(S\) does not affect the contribution of the player \(k\) once he has adhered to the coalition, the number of calculations needed for the
marginal values is \( \sum_{s=0}^{n-1} C^s_{n-1} = 2^{n-1} \). If we do not take into account this simplification, we can write the extended formula of the Shapley Value as follows:

\[
C_k = \frac{1}{n!} \sum_{i=1}^{n!} MV(\sigma^i, k) \tag{27}
\]

for every of the \( n! \) permutation or \( \sigma \) order, the players \( k \) have only one position that determine the coalition that can adhere. The term \( MV(\sigma^i, k) \) equals to the marginal value of adding the player \( k \) to its coalition.

The properties of the decomposition of this approach are:

- The symmetry, that The symmetry which ensures that the contribution of each factor is independent of its order of appearance on the list of the factors or the sequence.
- The additivity of components.

4.2 Decomposition of the Gini index according to the group

By supposing that groups represent factors which contribute to the Gini index of inequality, the component of group \( g \) according to the approach of Shapley is equal to what follows:

\[
C_g = \frac{1}{n!} \sum_{i=1}^{n!} MV(\sigma^i, g) \tag{28}
\]

where \( \sigma^i \) represents the \( ith \) possible order of groups and \( MV(\sigma^i, g) \) the impact of the elimination of the group \( g \) for the order \( \sigma^i \) to \( 6 \). A crucial stage to carry out this type of decomposition is well to determine the impact of the elimination of the factors (groups in our case) on the function, which is the index of Gini in our case. For better determining this idea, analyzing the impact of this determination on determining the group components of the statistic average. In this example, we will make the decomposition of the average \( \mu \) in components of two groups forming the population, that is to say A and B. The natural decomposition of the average is written in the following way:

\[\text{See the appendix C: for the decomposition of the statistics total according to the approach of Shapley}\]
\[ \mu = \frac{\phi_A \mu_A + \phi_B \mu_B}{C_A + C_B} \quad (29) \]

where \( \phi_g \) is the proportion of population of group \( g \). If we suppose that elimination of one factor - group - represents the case when simply we not take into account those households that compose the group, the decomposition according to the Shapley approach, is as follows\[Voir l’annexe A pour la décomposition selon l’approche de Shapley.\]

\[ C_{SA} = 0.5 \left[ \mu - \mu_B + \mu_A \right] \quad (30) \]

\[ C_{SB} = 0.5 \left[ \mu - \mu_A + \mu_B \right] \quad (31) \]

The condition necessary so that the two approaches are reconciling, such as \( C_A = C_{SA} \), is checked by what follows

\[ \frac{\mu_A}{\mu_B} = \frac{\phi_A}{\phi_B} \quad (32) \]

One can thus say, that a bad specification of the impact of the elimination of the factors on the function can lead us to a decomposition which is not quite founded.

If it is supposed now, that the elimination of the group \( g \) requires simply the subtraction of \( \phi_g \mu_g \), the equation\[29\] then represents a decomposition of Shapley such as the two approaches are reconciling.

While being based on the equation\[13\] the S-Gini index can be written in the following form:

\[ I_\rho = \sum_{i=1}^{N} p_{i,\rho} (1 - y_i / \mu) \quad (33) \]

In a first approach, we simply will make so that the weight \( p_{i,\rho} \) is equal to zero when it is supposed that the household \( i \) belongs to the group to be eliminated. This approach makes that the decomposition gives us a component by group equal to that defined by the equation\[16\]. To notice here that according to the equation\[16\] the sum of the group components explains the total of the inequality, and that the inequality will intra and intergroup are collected jointly by the group components.
4.3 Decomposition of the Gini index in inter and intragroup inequality

According to this decomposition we suppose, in the first stage, that the two factors are simply the intra and intergroup inequality. Hence, We express the total inequality as follows:

\[
I = C_{\text{inter}} + C_{\text{intra}} \tag{34}
\]

The rule that we will follow to collect the inequality in the presence of one of the two factors or other will be as follows:

- To eliminate the intragroup inequality and to calculate the intergroup inequality, \( I(\mu_1, ..., \mu_g) \), we will use a vector of income whose each household has the average income of its group is \( \mu_g \);

- To eliminate the intergroup inequality and to calculate the intragroup inequality, \( I(y_i(\mu/\mu_g)) \), we will use a vector of income whose each household has its income multiplied \( \mu/\mu_g \). That made that the average of the incomes of each group is equals to \( \mu \).

To eliminate the factor from arbitrary, that is to say to start by eliminating one or the other, and while being based on the Shapley approach, this decomposition gives us what follows:

\[
C_{\text{inter}} = 0.5 \left[ I - I(y_i(\mu/\mu_g)) + I(\mu_1, ..., \mu_G) \right] \tag{35}
\]

\[
C_{\text{intra}} = 0.5 \left[ I - I(\mu_1, ..., \mu_G) + I(y_i(\mu/\mu_g)) \right] \tag{36}
\]

Starting from this decomposition, one can make one second stage of decomposition, and that, by decomposing the intragroup component to specific group components. As we can notice it starting from the equation which defines the contribution of the intragroup inequality, this contribution is based on three indices of inequality. What we propose to do this decomposition is to base itself again on the Shapley approach to decompose the three terms.

\[
C_{\text{intra}} = 0.5 \left[ \underbrace{I}_{\text{terme1}} - \underbrace{I(\mu_A, \mu_B)}_{\text{terme2}} + \underbrace{I(y_i^A(\mu/\mu_A), y_i^B(\mu/\mu_B))}_{\text{terme3}} \right] \tag{37}
\]
It is said that the intragroup inequality is eliminated when the income of each household is equal to the average of its group. One this direction, we will observe the same rule for the three terms in the following way:

\[ C_A = \sum_{i=1}^{3} 0.25C_{A:terme(i)} \] (38)

\[ C_{A:terme1} = [I - I(\mu_A, y_B) + I(y_A, \mu_B) - I(\mu_A, \mu_B)] \]
\[ C_{A:terme2} = [I(\mu_A, \mu_B) - I(\mu_A, y_B) + I(\mu_A, \mu_B) - I(\mu_A, \mu_B)] = 0 \] (39)
\[ C_{A:terme3} = [I(y_i^A(\mu/\mu_A), y_i^B(\mu/\mu_B)) - I(\mu, y_i^B(\mu/\mu_B))] + [I(y_i^A(\mu/\mu_A), \mu) - I(\mu, \mu)] \]

5 Decomposition of the Gini index by sources of income

The decomposition of the index of Gini by sources of income is also interesting. That allows, to have a clear idea on the sources which contribute more to the inequality. First of all, it is supposed that the sum of \( K \) sources is equal to the total income of the household. The natural decomposition of the single parameterized Gini index can be done while being useful of the equations [12] and [13] in the following way:

\[ I_\rho = 1 - \sum_{i=1}^{N} P_{i,\rho} \sum_{k=1}^{K} s_{k,i} \] (40)
\[ = 1 - \sum_{k=1}^{K} \bar{\xi}(s_k) = \sum_{k=1}^{K} \left( \mu_k - \bar{\xi}(s_k) \right) \] (41)
\[ = \sum_{k=1}^{K} \frac{\mu_k}{\mu} C_\rho(s_k) \] (42)

\( C_\rho(s_k) \) is the coefficient of concentration of the source \( k \). Again, one can arrive easily at the same result expressed by the equation [40] when the index of Gini is written in the form of averages of expected relative deprivations such as:

\footnote{These decompositions are already programmed in the software DAD 4.3}
\[
\delta_i = \frac{\sum_{j=1}^{N} (\sum_{k=1}^{K} s_{k,j} - \sum_{k=1}^{K} s_{k,i}) +}{N} = \frac{\sum_{k=1}^{K} \sum_{j=1}^{N} (s_{k,j} - s_{k,i}) \cdot I(y_j > y_i)}{N} = \sum_{k=1}^{K} \bar{d}_{i,k} \quad (43)
\]

By using equation (43), we have then:

\[
I = \sum_{k=1}^{K} \frac{\sum_{i=1}^{N} \bar{d}_{i,k}}{N \mu} = \sum_{k=1}^{K} \frac{\mu_k}{\mu} C(s_k) \quad (46)
\]

According to the equation (43), one can notice that the expected relative deprivation of the household \(i\), \(\delta_i\), can be represented by the conditional sum of the deprivations relating by sources to its deprivation of income. Starting from this result, one can say that the contribution of a source is high if its coefficient of concentration is high. Generally, this is the case if the deprivation, for a given source, is very high for the poor of the population.

Without basing itself on the natural decomposition. One can also use the Shapley approach to estimate the contribution of each source. As we have to evoke it previously, a crucial stage to carry out this decomposition is to determine the impact of the elimination of a factor on the function of Gini index. What we propose will be to replace the source \(K\) by its average through the populations, that is to say \(\mu_k\), if this source or factor is eliminated. One can note that the natural decomposition and that according to the approach of Shapley give the same result in the case or each source gives the same classification of the households as that obtained starting from the levels of incomes. In this special case, the contribution of the source \(k\) is equal to what follows to also \(^9\):

\[
C_k = \frac{\mu_k}{\mu} I(s_k) \quad (47)
\]

This result is very simple to prove since the index of inequality is additively separable such as the elimination of one or more factors does not generate a reclassi-
\begin{equation}
I(s_1, s_2, \ldots, s_k|k \leq K) = \frac{\mu_k}{\mu} I(s_k)
\end{equation}

To also recall that when an index is additively separable, the algebraic or natural decomposition and its of Shapley give us the same result.

6 Absolute and relative contribution

We note the absolute contribution of each factor $k$ to the index of inequality of Gini by the component $CA_k$. This value gives us the importance, in absolute value, of the contribution of factor $k$. For the relative contribution of each factor to the index of Gini, we use the coefficient of relative contribution defines as follows:

\begin{equation}
CR_k = \frac{C_k}{I}
\end{equation}

7 Conclusion

References


ANNEX A: Binomial Theorem of Newton

What Newton discovered was a formula for \((a+b)^n\) that would work for all values of \(n\), including fractions and negatives:

\[
(a + b)^n = \sum_{s=0}^{n} C^s_n a^{n-1} b^s \quad \forall (a, b) \in \mathbb{R}, n \in \mathbb{N} \tag{A.1}
\]

Raising \((a+b)\) to the power \(n\) is equivalent to multiplying \(n\) identical binomials \((a+b)\). The result is a sum where every element is the product of \(n\) factors of type \(a\) or \(b\). The terms are thus of the form \(a^{n-p}b^p\). Each of these terms is obtained a number of times equal to \(C^p_n\), which is how many times we can choose \(p\) elements among \(n\). When \(a=b=1\), we will have:

\[
(1 + 1)^n = \sum_{s=0}^{n} C^s_n = 2^n \tag{A.2}
\]

Hence, we can conclude that:

\[
\sum_{s=0}^{n-1} C^s_n = 2^{n-1} \tag{A.3}
\]

ANNEX B: Decomposition of the total index according to the Shapley approach

When the marginal contribution of the factor \(k\), \(MV(S, k) = \bar{x}\), is constant for any order or coalition \(S\), the Shapley value of factor \(k\) is as follows:

\[
C_k = \sum_{\substack{s \subseteq S \setminus \{k\} \subseteq \{0,n-1\}}} \frac{s!(n-s-1)!}{n!} \bar{x} \tag{B.1}
\]
\[ C_k = \sum_{s=0}^{n-1} C_{n-1}^s \frac{s!(n-s-1)!}{s!n!} \bar{x} \]

\[ = \sum_{s=0}^{n-1} \frac{(n-1)!}{s!(n-s-1)!} \frac{s!(n-s-1)!}{n!} \bar{x} \]

\[ = \sum_{s=0}^{n-1} \frac{1}{n} \bar{x} = \bar{x} \] (B.2)

QED

ANNEXE C: Decomposition by sources

Let the two sources \( A \) and \( B \) that give the same order like the income. By using equations \( ??,33 \) we have what as follows:

\[
C_A = 0.5 \left[ \left( 1 - \frac{\xi_A + \xi_B}{\mu} \right) - \left( 1 - \frac{\mu_A + \xi_B}{\mu} \right) + \left( 1 - \frac{\xi_A + \mu_B}{\mu} \right) \right]
\]

\[ = 0.5 \left[ \frac{\mu_A + \mu_B}{\mu} - \frac{2\xi_A}{\mu} - \left( \mu_B - \mu_A \right) \right] \]

\[ = 0.5 \left[ \frac{2\mu_A}{\mu} - \frac{2\xi_A}{\mu} \right] \]

\[ = \frac{\mu_A}{\mu} G_A \] (C.1)