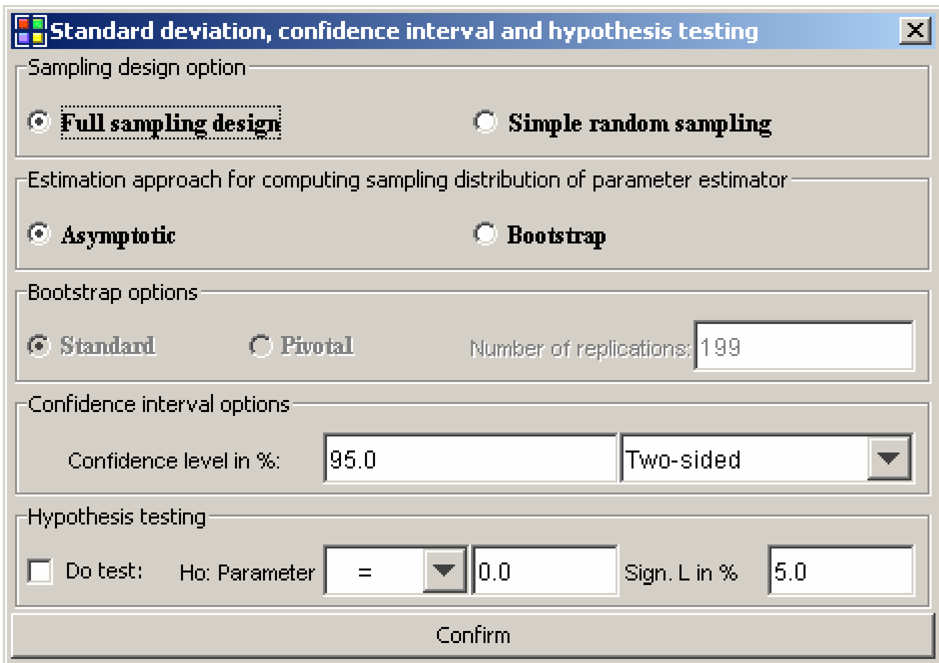
Standard deviation, confidence intervals and hypothesis testing

Starting with version 4.3 of DAD, one can, for some of the applications, compute confidence intervals and perform statistical tests by using asymptotic, standard bootstrap as well as pivotal bootstrap approaches. To see how, activate the following dialogue box (within an application window) by clicking on the button **“S.D. STD”**



After choosing the desired options, click on the button **“Confirm”** to confirm your choice.

OPTIONS

A) Sampling Design option;

One can choose between two categories of sampling design:

1) A broad and general one, activated through “Full sampling design”.
2) A simple one, activated through “Simple random sampling”.

For more information concerning this, see the section “Taking into account sampling design in DAD”

B) Approaches to estimating the sampling variability of DAD’s estimators;

DAD generally supports two approaches:

1) The asymptotic approach *(for many of the applications)*
2) The bootstrap approach. *(for some of the applications)*

C) Bootstrap options;

We can choose between two types of bootstrap options.

1) standard
2) and pivotal

We can also specify the number of bootstrap replications to be performed

D) Confidence Level;

Here, we can choose the:

1) confidence level (by default 95%) of our confidence intervals
2) and whether the confidence intervals should be "Two Sided ", "Lower Bounded" or "Upper Bounded".

E) Hypothesis testing;

We can carry out hypothesis testing by checking the box "Do test" and by inserting the appropriate values for the hypothesis test procedure.

ASYMPTOTIC APPROACH

Using the law of large numbers and the central limit theorem, it is possible to show that most of DAD's estimators ($\hat{\mu}$, say) of some distributive index μ are consistent and asymptotically normally distributed, with a sampling variance given by $\sigma^2_{\hat{\mu}}$. $\sigma^2_{\hat{\mu}}$ is (almost) always unknown, but we can generally estimate it consistently by $\hat{\sigma}^2_{\hat{\mu}}$ – and this is typically readily provided by DAD. Asymptotically, we can then write that

$$\hat{\mu} \sim N\left(\mu,\hat{\sigma}^2_{\hat{\mu}}\right)$$

which also implies that:

$$\frac{\hat{\mu}-\mu}{\hat{\sigma}_{\hat{\mu}}} \sim N(0,1)$$

Hypothesis testing and statistical decisions

The decision to reject or not some null hypothesis depends on the significance level α of the test. Let m be the value that $\hat{\mu}$ takes in a particular sample (the estimate of μ). The rejection rule can be described as follows:

Case a: a symmetric test

Reject $H_0{:}\mu=\mu_0$ in favor of $H_1{:}\mu \neq \mu_0$ if and only if: $\mu_0<m-\hat{\sigma}_{\hat{\mu}}z_{1-\alpha/2}$ or $\mu_0>m-\hat{\sigma}_{\hat{\mu}}z_{\alpha/2}$

This is because we have that

$$P\left(\mu_0+\hat{\sigma}_{\hat{\mu}} z_{\alpha/2}>\hat{\mu} \text{ or } \hat{\mu}>\mu_0+\hat{\sigma}_{\hat{\mu}} z_{1-\alpha/2}\right)=\alpha .$$

Note that this is equivalent to:

$$z_0<z_{\alpha/2} \text{ or } z_0>z_{1-\alpha/2} \text{ where } z_0=(m-\mu_0)/\hat{\sigma}_{\hat{\mu}}$$

Case b: testing an upper-bound null hypothesis

Reject $H_0:\mu \leq \mu_0$ in favour of $H_1:\mu>\mu_0$:

if and only if:

$\mu_0<m-\hat{\sigma}_{\hat{\mu}} z_{1-\alpha}$, which is equivalent to $z_0>z_{1-\alpha}$

Case c: testing a lower-bound null hypothesis test:

Reject $H_0:\mu \geq \mu_0$ in favour of $H_1:\mu<\mu_0$

if and only if :

$\mu_0>m-\hat{\sigma}_{\hat{\mu}} z_\alpha \Rightarrow z_0<z_\alpha$

The following table summarizes the confidence intervals and p-values corresponding to each of the three cases of the above hypothesis tests:

Case	Confidence interval	p – Value	Type
a	$[m-\hat{\sigma}_{\hat{\mu}} z_{1-\alpha/2}, m-\hat{\sigma}_{\hat{\mu}} z_{\alpha/2}]$	$2[1-F(\lvert z_0\rvert)]$	Two sided
b	$[m-\hat{\delta}_\mu z_{1-\alpha}, +\infty]$	$1-F(z_0)$	Lower-bounded confidence interval
c	$[-\infty, m-\hat{\delta}_\mu z_\alpha]$	$F(z_0)$	Upper-bounded confidence interval

STANDARD BOOTSTRAP APPROACH

Let the vector V regroup the ordered sample values of the estimator μ computed from B simulated or bootstrap samples, each drawn from the same initial sample. In the bootstrap approach, the vector V is the main tool to capture the distribution of the estimator μ. The number of replications B should be chosen so that $\alpha(B+1)$ is an integer and $B \geq (1-\alpha)/\alpha$. Let μ^*_α be the α -quantile of the vector V .

Once the significance level of the test is chosen, the rejection rule becomes:

$$\begin{aligned}
&a-\text{Reject } H_0: \quad \mu=\mu_0 \text{ vs } H_1:\mu\neq\mu_0 \quad \text{if}: \mu_0>\mu^*_{1-\alpha/2} \text{ or } \mu_0<\mu^*_{\alpha/2}\\
&b-\text{Reject } H_0: \quad \mu\leq\mu_0 \text{ vs } H_1:\mu<\mu_0 \quad \text{if}: \mu_0>\mu^*_{\alpha}\\
&c-\text{Reject } H_0: \quad \mu\geq\mu_0 \text{ vs } H_1:\mu>\mu_0 \quad \text{if}: \mu_0<\mu^*_{1-\alpha}
\end{aligned}$$

where μ_0 is the hypothesized value of μ. The following table summarizes the confidence intervals and p-values corresponding to the standard bootstrap approach:

Case	Confidence interval	p – Value	Type
a	$[\mu^*_{\alpha/2}, \mu^*_{1-\alpha/2}]$	$2\min(\sum_{i=1}^{B} I(\mu_i^* \le \mu_0), \sum_{i=1}^{B} I(\mu_i^* \ge \mu_0))/B$	Two sided
b	$[\mu^*_{\alpha}, +\infty]$	$\sum_{i=1}^{B} I(\mu_i^* \ge \mu_0)/B$	Lower-bounded confidence interval
c	$[-\infty, \mu^*_{1-\alpha}]$	$\sum_{i=1}^{B} I(\mu_i^* \le \mu_0)/B$	Upper-bounded confidence interval

PIVOTAL BOOTSTRAP APPROACH

Let the vector $\overline{V}$ be defined as

$$\overline{V}=\{t_1^*, t_2^*, \cdots, t_B^*\}$$

such that

$$t_i^* = \frac{\mu_i^* - \overline{\mu}}{\hat{\sigma}_i^*}$$

where $\overline{\mu}$ and $\hat{\delta}_i^*$ are respectively the average of the bootstrap μ_i^* and the standard deviation of the estimator estimated from the bootstrap samples . Let m be the value that $\hat{\mu}$ takes in a particular sample (the estimate of μ). The rejection rules are then:

a- Reject H_0: $\mu=\mu_0$ in favour of H_1:$\mu \neq \mu_0$ if: $\mu_0 < m-\hat{\sigma}_{\hat{\mu}} t^*_{1-\alpha/2}$ or $\mu_0 > \hat{\mu}-\hat{\sigma}_{\hat{\mu}} t^*_{\alpha/2}$

b- Reject H_0: $\mu \le \mu_0$ in favour of H_1:$\mu > \mu_0$ if: $\mu_0 < m-\hat{\sigma}_{\hat{\mu}} t^*_{1-\alpha}$

c- Reject H_0: $\mu \ge \mu_0$ in favour of H_1:$\mu < \mu_0$ if: $\mu_0 > m-\hat{\sigma}_{\hat{\mu}} t^*_{\alpha}$

The following table summarizes the confidence intervals and p-values according to the pivotal bootstrap approach:

Case	Confidence interval	p – Value	Type
a	$[m-\hat{\sigma}_{\mu} t^*_{1-\alpha/2}\ ,\ \mu-\hat{\sigma}_{\mu} t^*_{\alpha/2}]$	$2*\min(\sum_{i=1}^{B} I(t_i^* \le z_0), \sum_{i=1}^{B} I(t_i^* \ge z_0))/B$	Two sided
b	$[m-\hat{\sigma}_{\mu} t^*_{1-\alpha}, +\infty]$	$\sum_{i=1}^{B} I(t_i^* \ge z_0)/B$	Lower-bounded confidence interval
c	$[-\infty, m-\hat{\sigma}_{\mu} t^*_{\alpha}]$	$\sum_{i=1}^{B} I(t_i^* \le z_0)/B$	Upper-bounded confidence interval

where:

- B is the number of replications