An algorithm for computing the *Shapley Value*

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1 The *Shapley Value*

Consider a set N of n players that must divide a given surplus among themselves. The players may form coalitions (these are subsets S of N) that appropriate themselves a part of the surplus and redistribute it between their members. We denote the number of players in subset S by s. The function $v(S)$ is assumed to determine the coalition force, *i.e.*, which surplus can be divided without resorting to an agreement with the outsider players. The question to resolve is: How can the surplus be divided between the n players? According to the Shapley value, the surplus to be allocated to player k, denoted by e_k, is given by

$$e_k = \sum_{\substack{S \subset N \\ s \in \{0, n-1\}}} \frac{s!(n-s-1)!}{n!} mv(S,k) \tag{1}$$

where

$$mv(S,k) = \left(v\left(S \cup \{k\}\right) - v(S)\right). \tag{2}$$

The term $mv(S,k)$ is the marginal value that player k generates after his adhesion to a coalition S. Because the order of the players in a coalition S does not affect the contribution of the player k once he has adhered to the coalition, the number of calculations of $mv(S,k)$ needed in (1) is 2^n, since we need to compute $mv(S,k)$ with and without each of the n players.

2 A Shapley algorithm

The algorithm presented below explains how we may estimate the contribution of each player (or factor) in N to $v(N)$. It uses three steps.

Step 1: *Constructing a basic Shapley matrix* $\mathbf{B}$

Each row of $\mathbf{B}$ contains a distinct subset S_l of factors, $l \in \{1, 2, ..., 2^n\}$. The number of columns is n and that of rows is 2^n. Let $\rho_k = 2^{n-k}$.

To illustrate the algorithm, we set $n = 5$. The matrix $\mathbf{B}$ then looks like Figure 3. Each line l shows the factors that must be included in S_l for the computation of $v(S_l)$. The matrix $\mathbf{B}$ is self-explanatory. Its first row $S_1 = \{1\}$ says that v must first be computed only with factor 1; its second row says that v must then be computed with factors 1 and 2; and so on, until all possible subsets of N have been considered. For completeness, Steps **1.a-1.d** also provide a more detailed algorithm.

1.a: *Constructing the first column* c_1 of matrix $\mathbf{B}$.

- Initialize values of the first column to 1 in positions 1 to ρ_1. For instance, for $n = 5$, the first column takes the value 1 from level 1 to 16.
- Initialize values of the first column to 2 in $\rho_1 + 1$ to $\rho_1 + \rho_2$.
- Initialize values of the first column to k in $\sum_{j=1}^{k-1} \rho_j + 1$ to $\sum_{j=1}^{k} \rho_j$.
- Continue the initialization until $k = n$ (see values of the first column in Figure 1).

1.b: *Constructing the second column* c_2

- Copy values of c_1 in c_2 with the following structure:

from column c_1 *in:*			*to column* c_2 *in:*		
$\rho_1 + 1$	to	ρ_0	2	to	$\rho_1 + 1$
$\rho_1 + \rho_2 + 1$	to	ρ_0	$\rho_1 + 2$	to	$\rho_1 + \rho_2 + 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sum_{j=1}^{k} \rho_j + 1$	to	ρ_0	$\sum_{j=1}^{k-1} \rho_j + 2$	to	$\sum_{j=1}^{k} \rho_j + 1$

- Continue until $k = n$ (see column c_2 in Figure 1). Note the concordance of the values between the first and the second columns, emphasized in gray shading.

1.c: Constructing the remaining columns

- Use the same mapping of $c_1 \to c_2$ for $c_2 \to c_3$.
- Continue until $c_{n-1} \to c_n$. At the end of this process, a matrix similar to Figure 2 is obtained for $n = 5$.

1.d: Repositioning values

From the matrix obtained in ***1.c***, we generate the basic Shapley matrix $\mathbf{B}$, where each value is moved to its respective column at row level. For instance, the value of the fourth column in the sixth row equals 5 and must be moved to the fifth column in that same row. The new matrix is identical to Figure 3 for $n = 5$.

Step 2: *Constructing a Shapley matrix* $\mathbf{M}$ *of coefficients*

Denote the number of factors that appear in row l by g_l, and let $\bar{g}_l = n - g_l$. We generate two $2^n \times 1$ vectors $\mathbf{\Sigma}$ and $\bar{\mathbf{\Sigma}}$ and set their members respectively to $s_l = \max(0, g_l - 1)$ and $\bar{s}_l = \max(0, \bar{g}_l - 1)$ at position l (see Figure 4). Let the indicator $u_{l,k} = 1$ if factor $k \in S_l$ (the element $b_{l,k}$ of matrix $\mathbf{B}$ is a non-missing value) and zero otherwise. Let $\bar{u}_{l,k} = u_{l,k} - 1$. Elements of the matrix $\mathbf{M}$ are defined as

$$m_{l,k} = \frac{u_{l,k} s_l!(n - s_l - 1)! + \bar{u}_{l,k} \bar{s}_l!(n - \bar{s}_l - 1)!}{n!}. \tag{3}$$

Note that the matrices $\mathbf{B}$ and $\mathbf{M}$ can be computed without any knowledge of the function v.

Step 3: *Estimating the contributions of factors.*

3.1: Estimating the contribution of factor k according to the level of the introduction of that factor.

Let the $2^n \times 1$ vector $I_{k,z}$ be defined to estimate the average contribution of factor k once it is introduced at level $z \in [1, ..., n]$. Thus, $I_{1,2}$ will serve to estimate the second-level contribution of factor 1, obtained when factor 1 is introduced to compute the function $v(\cdot)$ after another factor has already been included. At position $l \in \{1, 2^n\}$, the element of $I_{k,z}$ equals $m_{l,k}$ if $u_{l,k} = 1$ and $g_l = z$ or if $\bar{u}_{l,k} = 1$ and $\bar{g}_l = z - 1$, and is set to zero otherwise. The contribution of factor k at level z is then equal to $I'_{k,z}\mathbf{V}$.

3.2: Estimating the elements of the Shapley decomposition.

Let the elements v_l of the $1 \times n$ vector $\mathbf{V} = (v_1, ..., v_{2^n})$ be given by function $v(S_l)$, where S_l is row l of matrix $\mathbf{B}$. Let the $1 \times n$ vector $\mathbf{E}$ be given by $\mathbf{E} = \mathbf{V}'\mathbf{M}$. Element e_k of $\mathbf{E}$ gives the Shapley contribution of factor k to $v(N)$, as defined also in (1).

Figure 1: Basic Shapley matrix $\mathbf{B}$

Level	c_1									c_2
1	1									.
2	1									2
3	1									2
4	1									2
5	1									2
6	1									2
7	1									2
8	1									2
9	1									2
10	1									3
11	1									3
12	1									3
13	1									3
14	1									4
15	1									4
16	1									5
17	2									.
18	2									3
19	2									3
20	2									3
21	2									3
22	2									4
23	2									4
24	2									5
25	3									.
26	3									4
27	3									4
28	3									5
29	4									.
30	4									5
31	5									.
32	.									.

Figure 2: Basic Shapley matrix $\mathbf{B}$: Steps 1.2 and 1.3

Level	c_1	c_2	c_3	c_4	c_5
1	1	.	.	.	.
2	1	2	.	.	.
3	1	2	3	.	.
4	1	2	3	4	.
5	1	2	3	4	5
6	1	2	3	5	.
7	1	2	4	.	.
8	1	2	4	5	.
9	1	2	5	.	.
10	1	3	.	.	.
11	1	3	4	.	.
12	1	3	4	5	.
13	1	3	5	.	.
14	1	4	.	.	.
15	1	4	5	.	.
16	1	5	.	.	.
17	2	.	.	.	.
18	2	3	.	.	.
19	2	3	4	.	.
20	2	3	4	5	.
21	2	3	5	.	.
22	2	4	.	.	.
23	2	4	5	.	.
24	2	5	.	.	.
25	3	.	.	.	.
26	3	4	.	.	.
27	3	4	5	.	.
28	3	5	.	.	.
29	4	.	.	.	.
30	4	5	.	.	.
31	5	.	.	.	.
32	.	.	.	.	.

Figure 3: Basic Shapley matrix $\mathbf{B}$: Step 1.4

Level	c_1	c_2	c_3	c_4	c_5
1	1	.	.	.	.
2	1	2	.	.	.
3	1	2	3	.	.
4	1	2	3	4	.
5	1	2	3	4	5
6	1	2	3	.	5
7	1	2	.	4	.
8	1	2	.	4	5
9	1	2	.	.	5
10	1	.	3	.	.
11	1	.	3	4	.
12	1	.	3	4	5
13	1	.	3	.	5
14	1	.	.	4	.
15	1	.	.	4	5
16	1	.	.	.	5
17	.	2	.	.	.
18	.	2	3	.	.
19	.	2	3	4	.
20	.	2	3	4	5
21	.	2	3	.	5
22	.	2	.	4	.
23	.	2	.	4	5
24	.	2	.	.	5
25	.	.	3	.	.
26	.	.	3	4	.
27	.	.	3	4	.
28	.	.	3	.	5
29	.	.	.	4	.
30	.	.	.	4	5
31	.	.	.	.	5
32	.	.	.	.	.

Figure 4: Size of coalitions without a potential factor

Level	Σ	$\bar{\Sigma}$
1	0	3
2	1	2
3	2	1
4	3	0
5	4	0
6	3	0
7	2	1
8	3	0
9	2	1
10	1	2
11	2	1
12	3	0
13	2	1
14	1	2
15	2	1
16	1	2
17	0	3
18	1	2
19	2	1
20	3	0
21	2	1
22	1	2
23	2	1
24	1	2
25	0	3
26	1	2
27	2	1
28	1	2
29	0	3
30	1	2
31	0	3
32	0	4

Figure 5: Matrix of coefficients $\mathbf{M}$ (Step 2)

Level	c_1	c_2	c_3	c_4	c_5
1	0.2000	-0.0500	-0.0500	-0.0500	-0.0500
2	0.0500	0.0500	-0.0333	-0.0333	-0.0333
3	0.0333	0.0333	0.0333	-0.0500	-0.0500
4	0.0500	0.0500	0.0500	0.0500	-0.2000
5	0.2000	0.2000	0.2000	0.2000	0.2000
6	0.0500	0.0500	0.0500	-0.2000	0.0500
7	0.0333	0.0333	-0.0500	0.0333	-0.0500
8	0.0500	0.0500	-0.2000	0.0500	0.0500
9	0.0333	0.0333	-0.0500	-0.0500	0.0333
10	0.0500	-0.0333	0.0500	-0.0333	-0.0333
11	0.0333	-0.0500	0.0333	0.0333	-0.0500
12	0.0500	-0.2000	0.0500	0.0500	0.0500
13	0.0333	-0.0500	0.0333	-0.0500	0.0333
14	0.0500	-0.0333	-0.0333	0.0500	-0.0333
15	0.0333	-0.0500	-0.0500	0.0333	0.0333
16	0.0500	-0.0333	-0.0333	-0.0333	0.0500
17	-0.0500	0.2000	-0.0500	-0.0500	-0.0500
18	-0.0333	0.0500	0.0500	-0.0333	-0.0333
19	-0.0500	0.0333	0.0333	0.0333	-0.0500
20	-0.2000	0.0500	0.0500	0.0500	0.0500
21	-0.0500	0.0333	0.0333	-0.0500	0.0333
22	-0.0333	0.0500	-0.0333	0.0500	-0.0333
23	-0.0500	0.0333	-0.0500	0.0333	0.0333
24	-0.0333	0.0500	-0.0333	-0.0333	0.0500
25	-0.0500	-0.0500	0.2000	-0.0500	-0.0500
26	-0.0333	-0.0333	0.0500	0.0500	-0.0333
27	-0.0500	-0.0500	0.0333	0.0333	0.0333
28	-0.0333	-0.0333	0.0500	-0.0333	0.0500
29	-0.0500	-0.0500	-0.0500	0.2000	-0.0500
30	-0.0333	-0.0333	-0.0333	0.0500	0.0500
31	-0.0500	-0.0500	-0.0500	-0.0500	0.2000
32	-0.2000	-0.2000	-0.2000	-0.2000	-0.2000

Figure 6: Vector $\mathbf{V}$ (Step 3)

Level	$\mathbf{V}$
1	$v(S_1)$
2	.
3	.
4	.
5	.
6	.
7	.
8	.
9	.
10	.
11	.
12	.
13	.
14	.
15	.
16	.
17	$v(S_l)$
18	.
19	.
20	.
21	.
22	.
23	.
24	.
25	.
26	.
27	.
28	.
29	.
30	.
31	.
32	$v(S_{2^5})$