# The Absolute Gini Coefficient: Decomposability and Stochastic Dominance

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# 1 The Absolute Gini Coefficient

As generally rule, the Absolute Gini Coefficient, $AI$ can be defined by what follows:

$$AI = I * \mu \tag{1}$$

where $I$ is Gini coefficient and $\mu$ the average of incomes that is supposed to be greater than zero. When $\mu <= 0$, one can use the general definition of the $AI$, presented in the following section.

# 2 The Gini coefficient and relative deprivation

According to Runciman (1966), the magnitude of relative deprivation is the difference between the desired situation and the actual situation of a person. We define the relative deprivation of household $i$ compared to $j$ as follows [1]:

$$\delta_{i,j} = (y_j - y_i)_+ = \begin{cases} y_j - y_i & if \quad y_i < y_j \\ 0 & otherwise. \end{cases} \tag{2}$$

where $y_k$ is the income for household $k$. The expected deprivation of household $i$ equals to:

$$\bar{\delta}_i = \frac{\sum_{j=1}^{N} (y_j - y_i)_+}{N} \tag{3}$$

where $N$ is the is population size. The $AI$ can be written in the following form:

$$AI = \sum_{i=1}^{N} \frac{\bar{\delta}_i}{N} = \bar{\delta} \tag{4}$$

This functional form of the Absolute Gini coefficient shows that this coefficient is just the average expected relative deprivation $\bar{\delta}$.

[1] See also Yitzhaki (1979) and Hey and Lambert (1980).

## 3 Absolute Gini coefficient and inequality axioms

Among the most useful properties that the $AI$ respects, one can report what follows:

*Symmetry axiom*: The $AI$ index does not depends on others characteristics of the individual in the exception of its income.

*Population axiom*: Inequality for k identical populations equals to the inequality of the population (easily to proof using equations (3-5)).

*Pigou-Dalton transfert axiom*: The transfer of a marginal amount from a richer person to a poorer one decreases inequality.

*Invariance to constant adding axiom*: Adding the same amount to all households does not affect inequality (easily to proof using equations (3-5)).

*Constance to proportional adding axiom*: Adding to each household a proportion of its income, noted by $\lambda$, increases the inequality by the same proportion, $AI((1+\lambda)y) = (1+\lambda)AI(y)$ (easily to proof using equations (3-5)).

*Axiomatic consistence*: All axioms are consistent and continue to be valid when the average of incomes is negative or equals to zero.

## 4 Decomposing the Absolute Gini Coefficient across groups

Starting from equation (3), the contribution of each household to total inequality depends on its expected relative deprivation. When household $k$ belongs to group $g$, one can rewrite its average relative deprivation as follows:

$$\bar{\delta}_k = \phi_g \bar{\delta}_{k,g} + \tilde{\delta}_{k,g} \tag{5}$$

$$\tilde{\delta}_{k,g} = \sum_{\substack{j=1 \\ j \notin g}}^{N-K_g} \frac{(y_k - y_j)_+}{N} \tag{6}$$

where $\phi_g$ is the population's share of group $g$, $K_g$ is the number of households that belong to the group $g$, $\bar{\delta}_{k,g}$ is the expected relative deprivation of household $k$ at the level of group $g$ and $\tilde{\delta}_{k,g}$ is the expected relative deprivation of household $k$ at the level of its complement group. By rewriting the Gini coefficient, we find:

$$\begin{aligned} AI &= \sum_{g=1}^{G} \sum_{k=1}^{K_g} \left[ \frac{\phi_g \bar{\delta}_{k,g} + \tilde{\delta}_{k,g}}{N} \right] && (7) \\ &= \sum_{g=1}^{G} \left[ \phi_g^2 \frac{\sum_{k=1}^{K_g} \bar{\delta}_{k,g}}{K_g} \right] + \sum_{g=1}^{G} \sum_{k=1}^{K_g} \frac{\tilde{\delta}_k}{N} && (8) \\ &= \sum_{g=1}^{G} \phi^2 AI_g + \widetilde{AI} && (9) \end{aligned}$$

where $G$ is the number of groups and $\widetilde{AI}$ is equal to the Absolute Gini coefficient where the relative deprivation within the group is ignored. With the group comparison, intergroup inequality is defined as the inequality when each household have the average income of its group. In such case, the decomposition of the $IA$ index takes the following form:

$$AI = \sum_{g=1}^{G} \phi^2 AI_g + AI(\mu_g) + R \tag{10}$$

Without group income overlap, the residue $R = \widetilde{AI} - AI(\mu_g)$ equals to zero[2].

[2] In the case of distribution without overlap, the relative deprivation of a given member of the poor group compared to other $m$ members of the rich group is equivalent to the $m$ differences between mean of the rich group and its income.

# 5 Decomposition of the Absolute Gini Coefficient by Income Components

The decomposition of the absolute Gini coefficient by income components is also interesting. This decomposition allows to have a clear idea on how each component contributes to the total inequality. First, one supposes that the sum of $K$ components equals the total income and the amount of component $k$, noted by $s_k$. Basing on equation (3) one can writes what follows:

$$\bar{\delta}_i = \frac{\sum_{j=1}^{N} \left(\sum_{k=1}^{K} s_{k,j} - \sum_{k=1}^{K} s_{k,i}\right)_+}{N} \tag{11}$$

$$= \sum_{k=1}^{K} \frac{\sum_{j=1}^{N} (s_{k,j} - s_{k,i}) * I(y_j > y_i)}{N} = \sum_{k=1}^{K} \bar{d}_{i,k} \tag{12}$$

where $I(y_j > y_i) = 1$ if $y_j > y_i$ and $0$ otherwise. By using equation (4), we have that:

$$AI = \sum_{k=1}^{K} \frac{\sum_{i=1}^{N} \bar{d}_{i,k}}{N\mu} = \sum_{k=1}^{K} AC_k \tag{13}$$

where $AC_k$ is the absolute concentration index of component $k$.

## 5.1 Showing the ranking effect

Starting from equation (13), the use of the concentration coefficient instead of the Gini coefficient for each component is implied by the interaction effect between components. To clarify this, one can write what follows:

$$AI = \sum_{k=1}^{K} [AI_k + (AC_k - AI_k)] \tag{14}$$

In the case where each component gives the same rank of households as total income, the ranking effect $(AC_k - AI_k)$ equals zero and we can write:

$$AI = \sum_{k=1}^{K} AI_k \tag{15}$$

Generally, the importance of the interaction effect can be estimated by the ratio, $IE = \frac{\sum_k |AC_k - AI_k|}{AI}$.

## 5.2 Interpreting the marginal contribution of income components

At this stage, we propose to shed light again on the marginal contribution of each component to the absolute Gini coefficient. For this purpose, we assume that the marginal contribution represents the variation in the absolute Gini coefficient implied by adding the $k_{th}$ component to the set of complement components. Based on equation (13), one can write:

$$AI - \overline{AI}_k = \Delta_k = (\overline{AC}_k - \overline{AI}_k) + (AC_k - \overline{AI}_k) \tag{16}$$

where:

- $\overline{AI}_k$: Absolute Gini coefficient excluding component $k$.
- $\overline{AC}_k$: Absolute Concentration coefficient excluding component $k$.

This explicit form gives us more information on the nature of each contribution. To better exhibit the advantages of this form, the following cases are presented.

*Proportional component*: If component $k$ represents the same proportion $\lambda$ of income for all households, then:

$$\Delta_k = \lambda \overline{AI}_k \tag{17}$$

*Ranked component*: If component $k$ has the same power of ranking households as the complement income, then:

$$\Delta_k = (AI_k - \overline{AI}_k) \tag{18}$$

*Non-ranked component*: This is the usual case. To check the direction of the impact, we can write the following condition:

$$\Delta_k > 0 \Rightarrow \frac{(AC_k - \overline{AI}_k)}{(\overline{AI}_k - \overline{AC}_k)} > 1 \tag{19}$$

If the main part of the interaction effect with the complement, expressed by $(\overline{AI}_k - \overline{AC}_k)$ in equation (16), is less important, the difference $(AC_k - \overline{AI}_k)$ determines the importance of the impact.

# 6 Absolute Inequality Dominance

## 6.1 The Absolute Lorenz Curve

One can recall here that the generalized Lorenz curves can be used directly to check the dominance in inequality as well as in social welfare when distributions have the same average income. For the absolute inequality indices, adding a constant to fill this restriction does not affect the level of inequality. Basing on the Lorenz curve approach to check the inequality dominance, one can see that the absolute inequality in distribution $A$ dominates that of $B$ if and only if [3]:

$$AL_A(p) < AL_B(p) \;\; \forall p \in [0,1] \tag{20}$$

where $AL_D$ is the absolute Lorenz curve for the distribution $D$, such that:

$$\begin{aligned} AL_D(p) &= \int_0^p (y_D(q) - \mu_D)dq && (21) \\ &= GL_D(p) - p\mu_D && (22) \end{aligned}$$

where $GL_D$ is the generalised Lorenz curve for the distribution $D$[4].

## 6.2 The Absolute Deprivation Curve

We begin by defining the link between the deprivation curve and the $AI$ index in continues form such that:

$$AI(p) = \int_0^1 \delta(p)dp \tag{23}$$

In discrete form and when incomes are ranked in from the lower to the higher, the generalized deprivation curve is defined $p_i = i/N$ by:

$$\delta(p = i/N) = \frac{\sum_{i=1}^{N} y_i - \sum_{j=1}^{i}(N-j+1)y_j}{N} \tag{24}$$

Starting from equation (24), one can write what follows [5]:

[3] See Atkinson (1970), Shorrocks (1983) and Kakwani (1984).

[4] Moyes (1987) and Jenkins (1991)

[5] If we normalize the absolute depravation by the average income, we find that: $\bar{\delta}(p) = 1 - L(p) - (1-p)L'(p)$.

$$\mu - \delta(p) = GL(p) + (1-p)Q(p) \tag{25}$$

$$\delta(p) = -AL(p) - (1-p)Q(p) \tag{26}$$

**Lemma 1** $AL_A(p) < AL_B(p) \not\Leftarrow \delta_A(p) > \delta_B(p)$.

# 7 Conclusion

In progress.

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